Linear independence and Locally Refined B-splines

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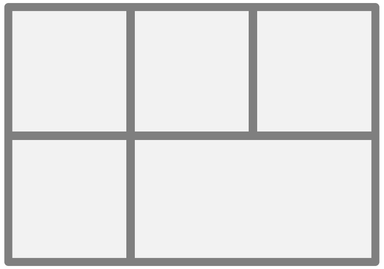
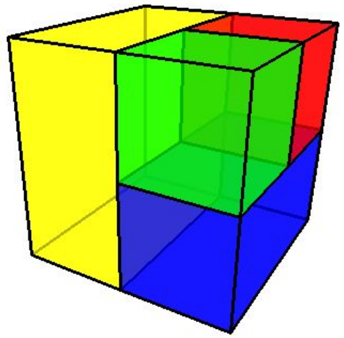
http://www.sintef.no/Projectweb/Computational-Geometry/

Spline space over Box-partitions

- LR B-splines, T-splines (as originally defined) and Hierarchal B-splines can all be regarded as splines defined over box-partitions.

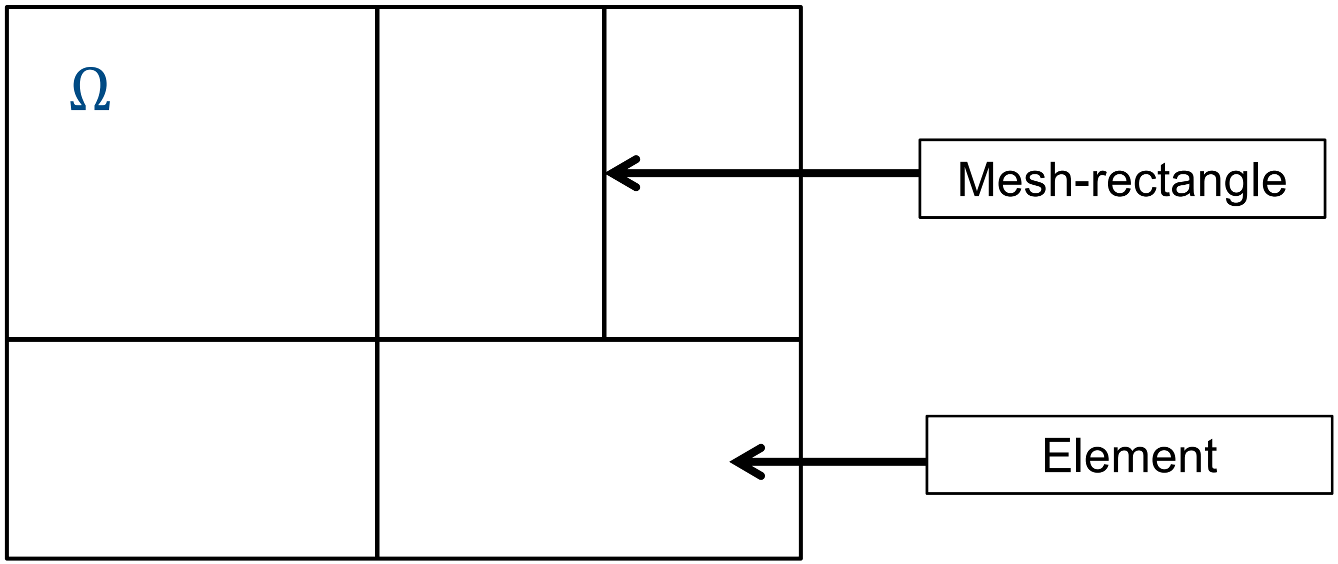




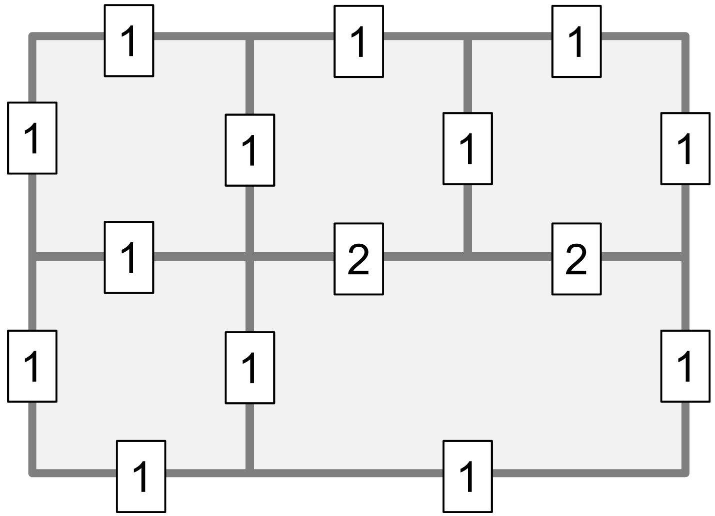
  - Hierarchical B-spline by multi-level mid-element refinement, with possible restriction of refinements to certain regions
  - T-splines by what is allowed by the T-spline refinement rules
  - LR-splines by a sequence of local refinements starting from a tensor product grid
    - introducing additional B-splines is specified regions as required

Box-partition

- $\Omega \subseteq \mathbb{R}^d$ a d-box in $\mathbb{R}^d$.
- A finite collection $\mathcal{E}$ of d-boxes in $\mathbb{R}^d$ is said to be a **box partition** of Ω if
  1. $\beta_1^o \cap \beta_2^o = \emptyset$ for any $\beta_1^o, \beta_2^o \in \mathcal{E}$, where $\beta_1^o \neq \beta_2^o$.
  2. $\cup_{\beta \in \mathcal{E}} \beta = \Omega$.

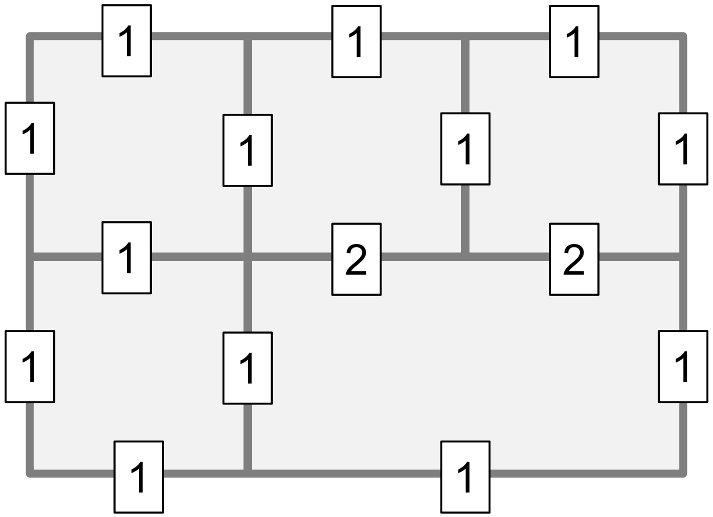


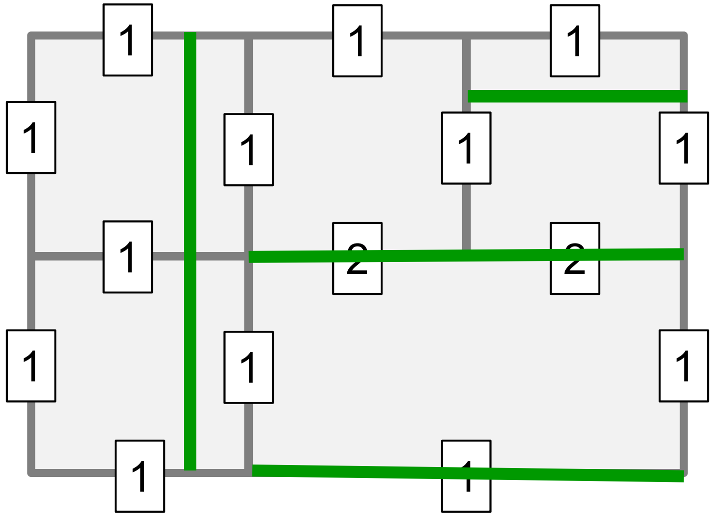
μ-extended box-mesh (adding multiplicities)



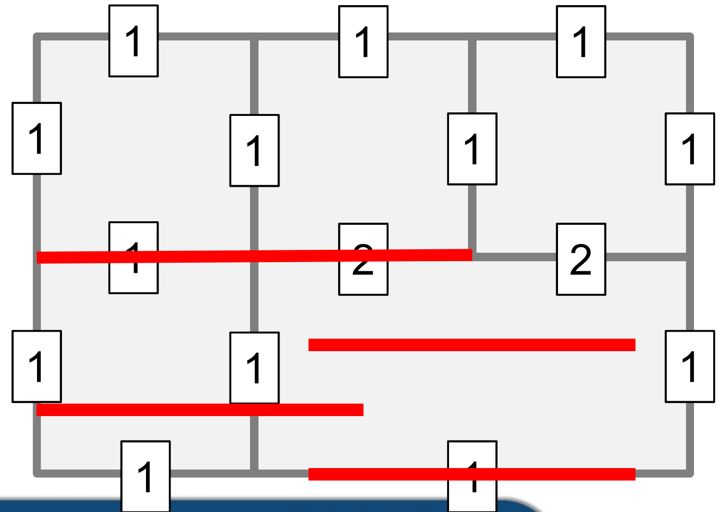
- A multiplicity μ is assigned to each mesh-rectangle
- Supports variable knot multiplicity for Locally Refined B-splines.

Refinement by inserting mesh-rectangles giving a constant split





Constant split

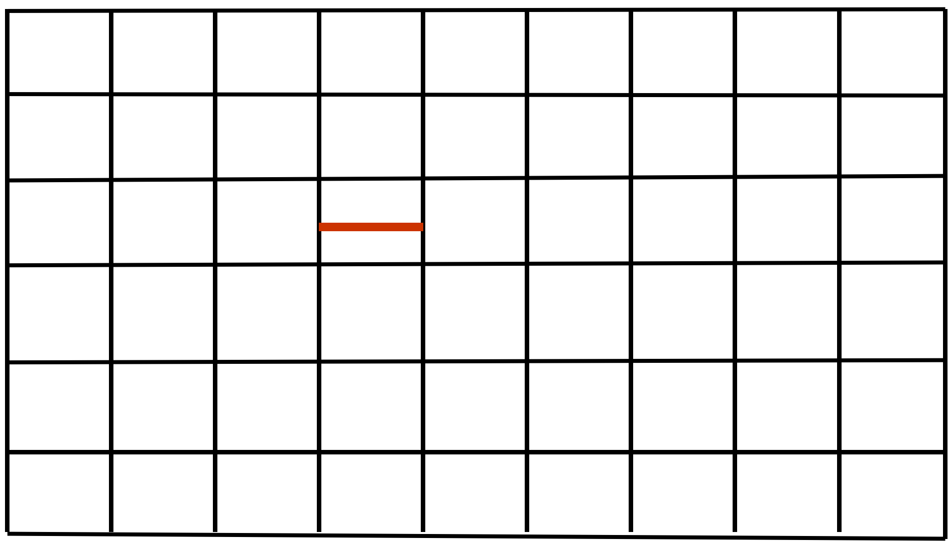


Not constant split

μ-extended LR-mesh

A μ -extended LR-mesh is a μ-extended box-mesh $(\mathcal{M}, \mu)$ where either

1. $(\mathcal{M}, \mu)$ is a tensor-mesh with knot multiplicities or
2. $(\mathcal{M}, \mu) = (\widetilde{\mathcal{M}} + \gamma, \tilde{\mu}_\gamma)$ where $(\widetilde{\mathcal{M}}, \tilde{\mu})$ is a μ -extended LR-mesh and γ is a constant split of $(\widetilde{\mathcal{M}}, \tilde{\mu})$.



All multiplicities not shown are 1.

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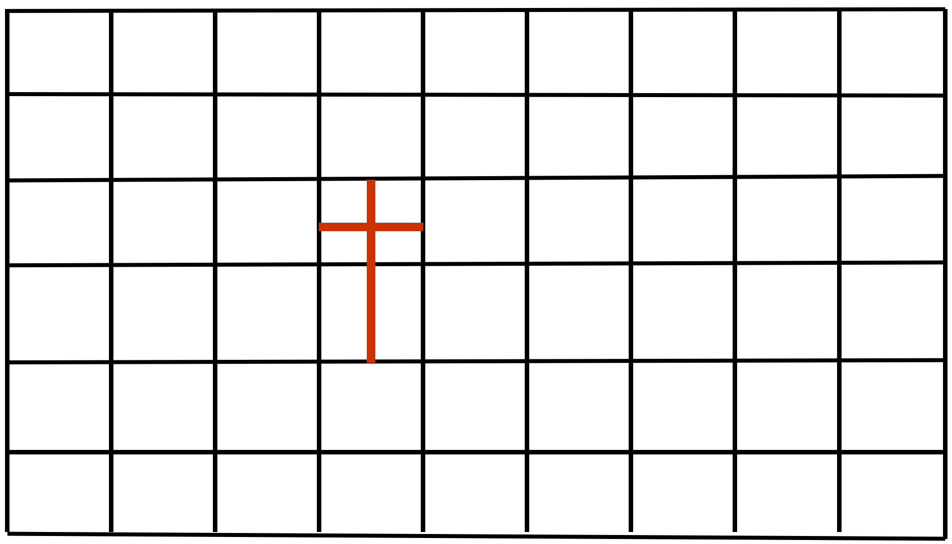

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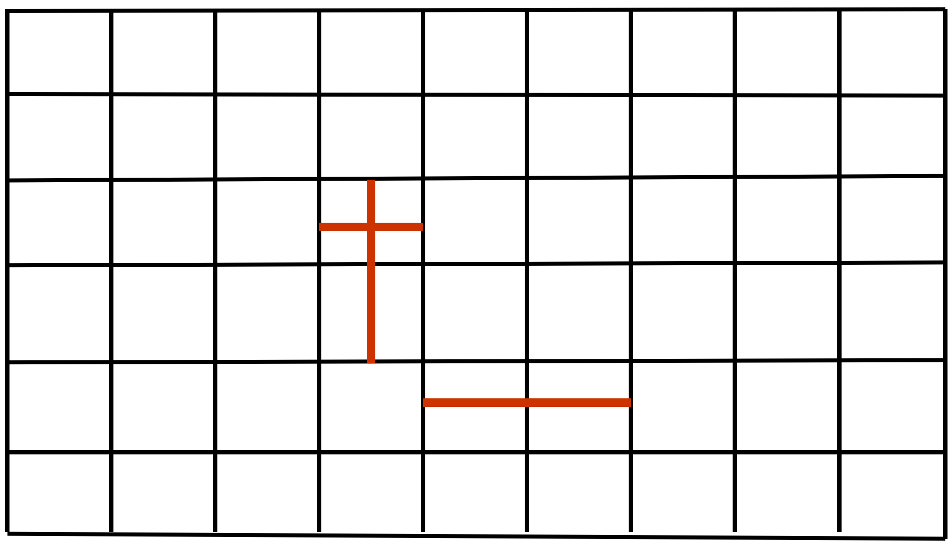

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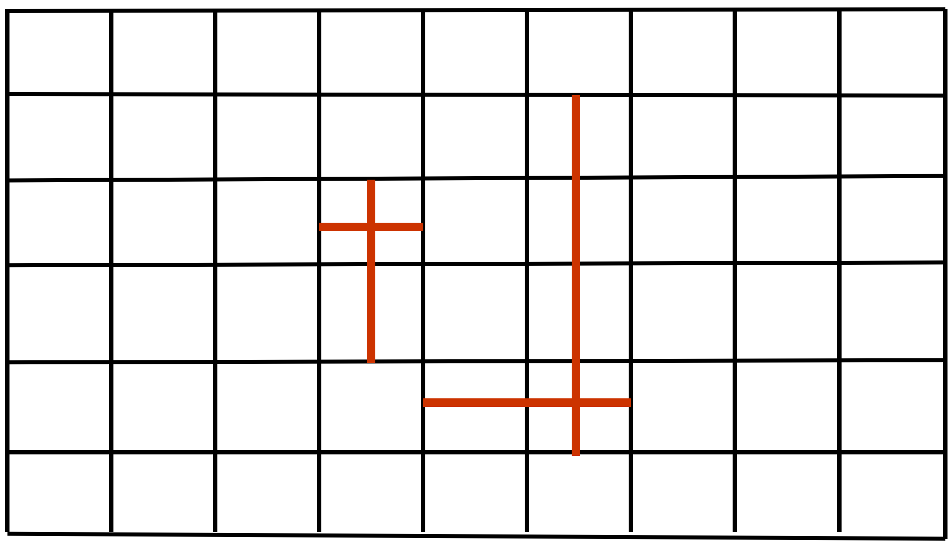

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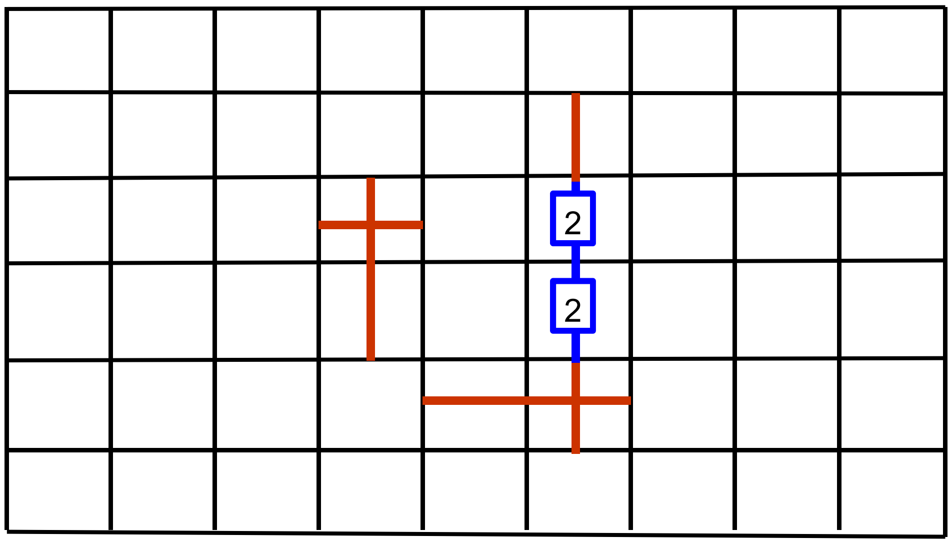

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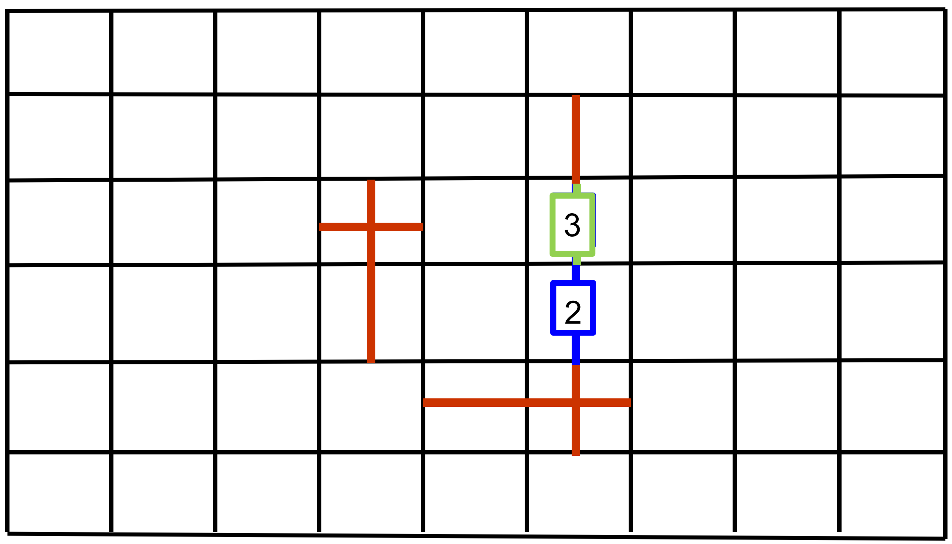

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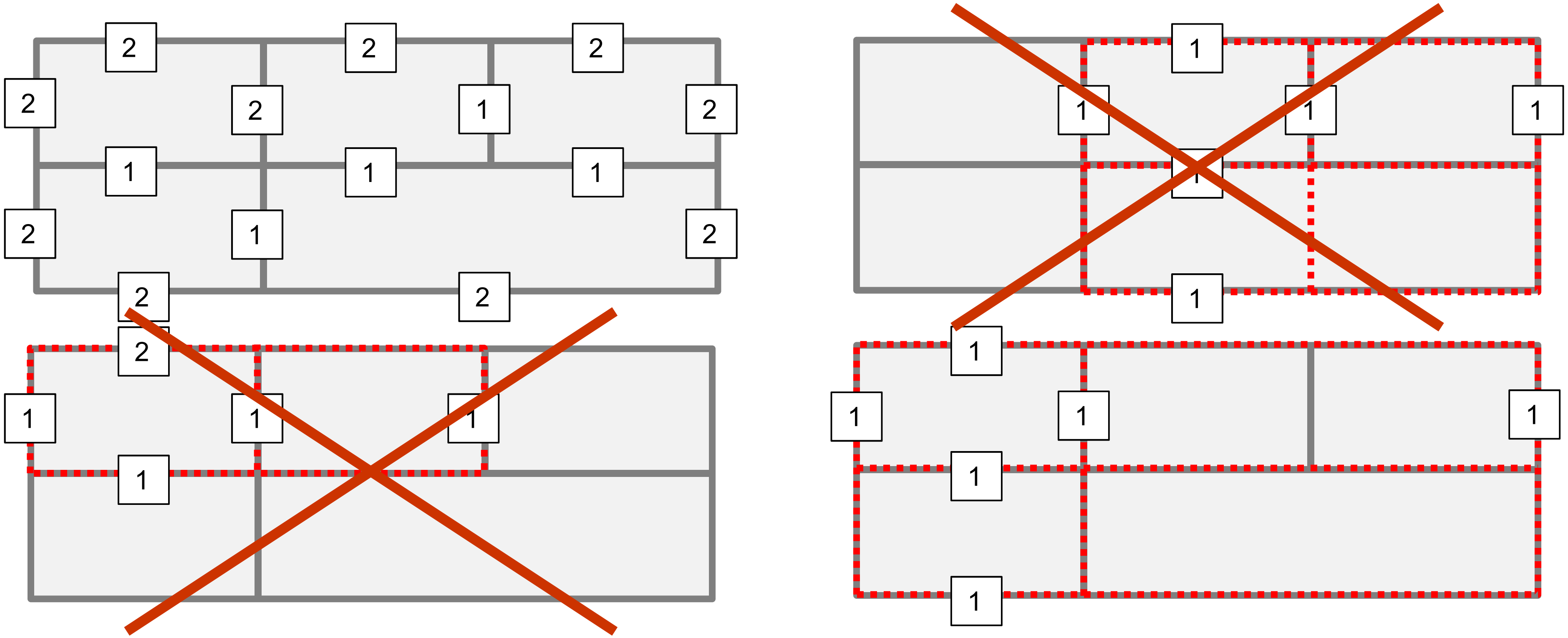
LR B-spline

Let (M, μ) be an μ-extended LR-mesh in $\mathbb{R}^d$. A function $B\colon \mathbb{R}^d \to \mathbb{R}$ is called an LR B-spline of degree $\boldsymbol{p}$ on $(\mathcal{M}, \mu)$ if B is a tensor-product B-spline with minimal support in $(\mathcal{M}, \mu)$.



Splines on a μ-extended LR-mesh

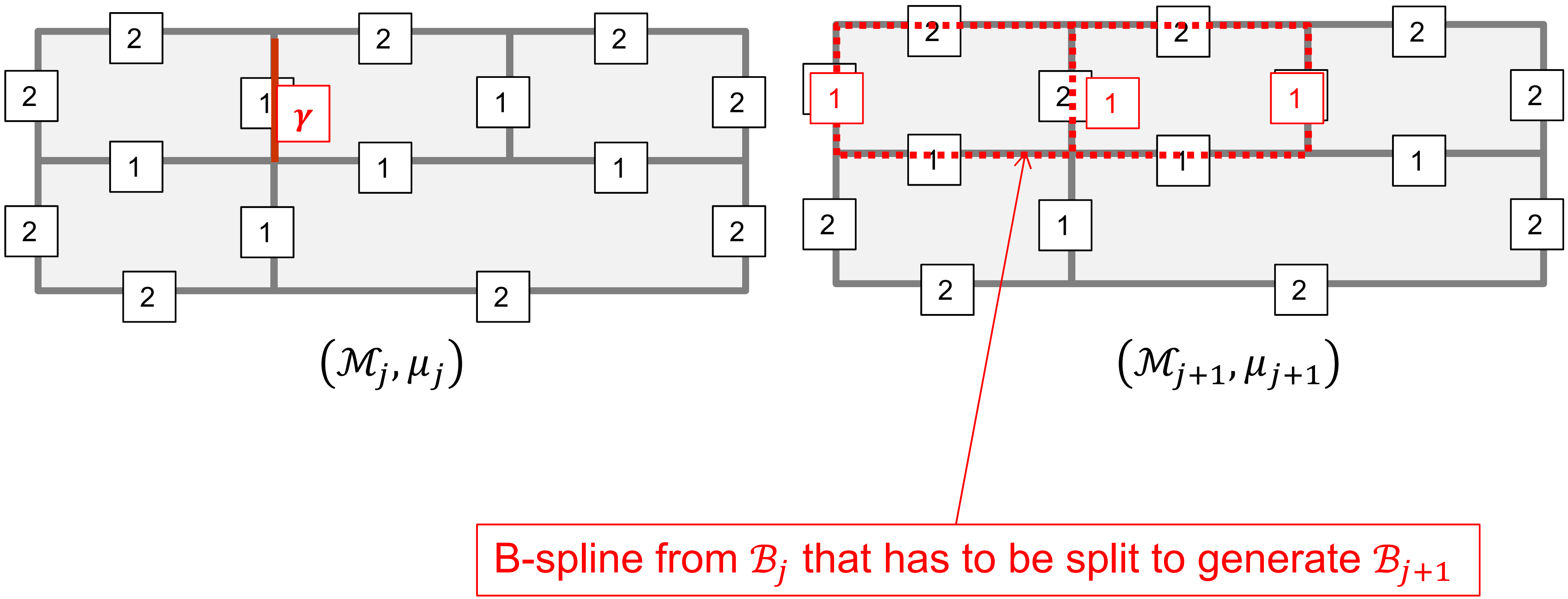
We define a sequence of $\boldsymbol{\mu}$-extended LR-meshes $(\mathcal{M}_1, \mu_1), \ldots, (\mathcal{M}_q, \mu_q)$ with corresponding collections of minimal support B-splines $\mathcal{B}_1, \ldots, \mathcal{B}_q$.

For $j = 1, \ldots, q-1$ creating $(\mathcal{M}_{j+1}, \mu_{j+1}) = (\mathcal{M}_j + \gamma_j, \mu_{j,\gamma_j})$ from $(\mathcal{M}_j, \mu_j)$ involves inserting a mesh-rectangles γ_j that increases the number of B-splines. More specifically:

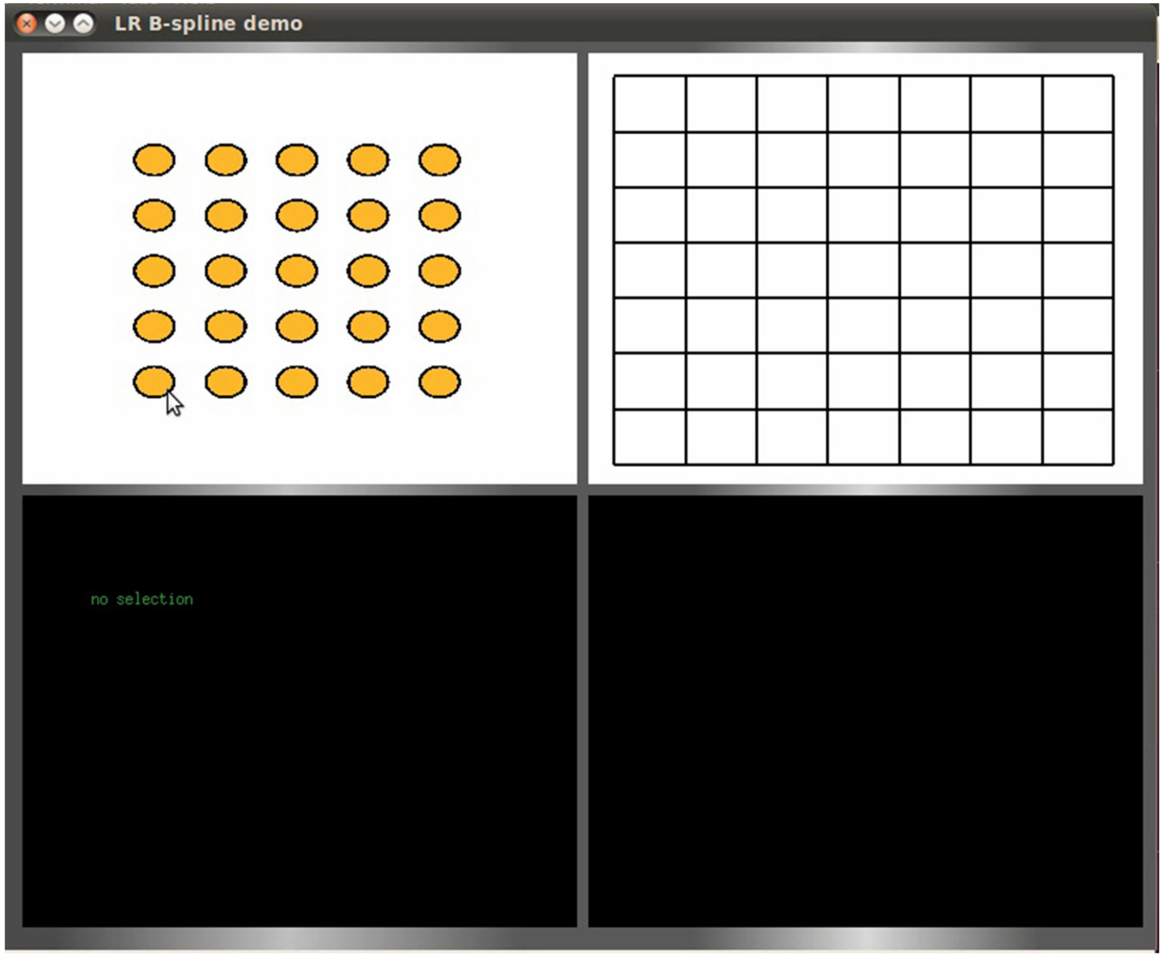
- γ_j splits $(\mathcal{M}_j, \mu_j)$ in a constant split.
- at least on B-spline in $\mathcal{B}_j$ does not have minimal support in $(\mathcal{M}_{j+1}, \mu_{j+1})$.

After inserting γ_j we start a process to generate a collection of minimal support B-splines $\mathcal{B}_{j+1}$ over $(\mathcal{M}_{j+1}, \mu_{j+1})$ from $\mathcal{B}_j$.

Going from $(\mathcal{M}_j, \mu_j)$ to $(\mathcal{M}_{j+1}, \mu_{j+1})$



Example LR B-spline refinement



Video by PhD fellow Kjetil A. Johannessen, NTNU, Trondheim, Norway.

Ensuring linear independence

- We say that $(\mathcal{M}_{j+1}, \mu_{j+1}, \boldsymbol{p})$ goes hand in hand with $(\mathcal{M}_j, \mu_j, \boldsymbol{p})$ if
  - $\text{span}\,(B)_{B \in \mathcal{B}_j} = \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}_j, \mu_j)$ and
  - $\text{span}\,(B)_{B \in \mathcal{B}_{j+1}} = \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}_{j+1}, \mu_{j+1})$.
- If $(\mathcal{M}_{j+1}, \mu_{j+1}, \boldsymbol{p})$ and $(\mathcal{M}_{j+1}, \mu_{j+1}, \boldsymbol{p})$ goes hand-in-hand and $\#\mathcal{B}_{j+1} = \dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}_{j+1}, \mu_{j+1})$ then the B-splines of $\mathcal{B}_{j+1}$ form a basis for $\mathbb{S}_{\boldsymbol{p}}(\mathcal{M}_{j+1}, \mu_{j+1})$.

To ensure linear independence we have to

1. Determine $\dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}_{j+1}, \mu_{j+1})$
2. Determine if $\mathcal{B}_{j+1}$ spans $\mathbb{S}_{\boldsymbol{p}}(\mathcal{M}_{j+1}, \mu_{j+1})$
3. Check that $\#\mathcal{B}_{j+1} = \dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}_{j+1}, \mu_{j+1})$

How to measure dimensional of spline space of degree p over a μ-extended box partition $(\mathcal{M}, \mu)$.

- Dimension formula developed (Mourrain, Pettersen)

$$\dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}, \boldsymbol{\mu}) = \sum_{\ell=0}^{d} (-1)^{d-\ell} \left(\sum_{\beta \in \mathcal{F}_\ell(\mathcal{M})} \prod_{k=1}^{d} \left(p_k - \mu_k(\beta) + 1 \right) \right) - \sum_{q=0}^{d-1} (-1)^{d-q} \dim H_q(\widetilde{\mathfrak{S}}(\mathcal{N}))$$

Combinatorial values calculated from topological structure

Homology terms

- In the case of 2-variate LR-splines always zero

Difference in spanning properties between $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$

- The only B-splines in $\mathcal{B}_{j+1}$ that model the discontinuity introduced by γ_j are those that have γ_j with multiplicity $\mu(\gamma_j)$ as part of the knot structure.
- By restricting these B-splines to γ_j we get a set of B-splines $\mathcal{B}_\gamma$ restricted to γ_j with dimension one lower than the dimension of the B-splines of $\mathcal{B}_{j+1}$.
- A theorem for general dimensions and degrees states

$$\dim \operatorname{span} \left(B_\gamma\right)_{B\in\mathcal{B}_\gamma} \leq \dim \mathbb{S}_{\boldsymbol{p}}\left(\mathcal{M}_{j+1}, \mu_{j+1}\right) - \dim \mathbb{S}_{\boldsymbol{p}}\left(\mathcal{M}_j, \mu_j\right)$$

- Further it is stated that $\mathcal{B}_{j+1}$ spans $\mathbb{S}_{\boldsymbol{p}}\left(\mathcal{M}_{j+1}, \mu_{j+1}\right)$ if

$$\dim \operatorname{span} \left(B_\gamma\right)_{B\in\mathcal{B}_\gamma} = \dim \mathbb{S}_{\boldsymbol{p}}\left(\mathcal{M}_{j+1}, \mu_{j+1}\right) - \dim \mathbb{S}_{\boldsymbol{p}}\left(\mathcal{M}_j, \mu_j\right)$$

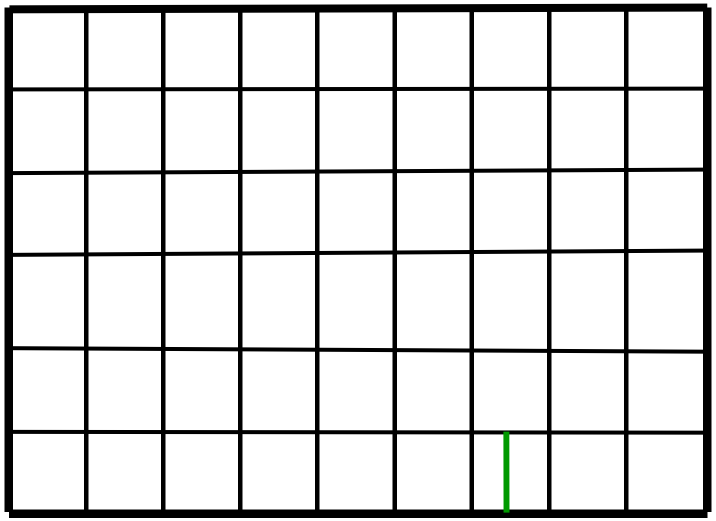
Observations

- To find the dimension of a spline space with many B-splines is more complex than finding the dimension of a spline space with few B-splines
- When assessing the B-splines $\mathcal{B}_\gamma$ over γ_j we first ensure that the refinement is broken into a sequence of LR B-spline refinements with as low dimension increase as possible.
  - As a legal LR-spline refinement always introduces at least one B-spline linearly independent from the pre-existing, a dimension increase by just one will ensure that we go hand-in-hand.
  - If the dimension increase is greater than 1 we need to assess the B-splines $\mathcal{B}_\gamma$ over γ_j.

Example: C^2 bi-cubic refinement configurations

Dimension increase of spline space over the box-partition

- All boundary knots mesh-rectangles have multiplicity 4
- All interior mesh-rectangles have multiplicity 1



Mesh-rectangle length 1 starting at the boundary, T-joint at other end. Dimension increase 1

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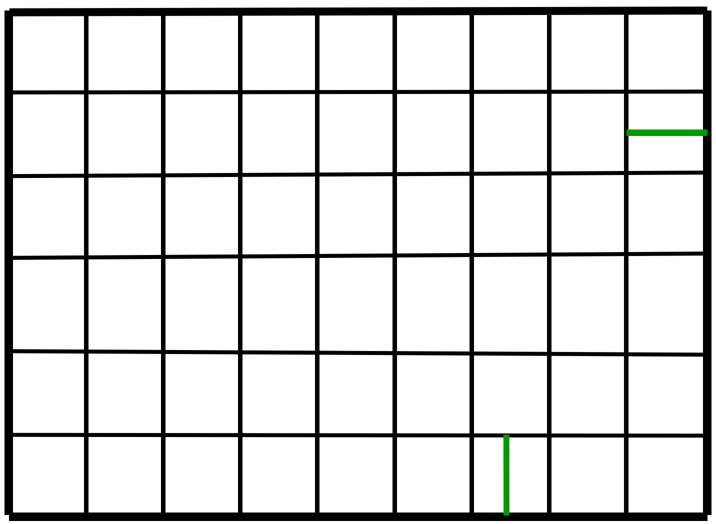

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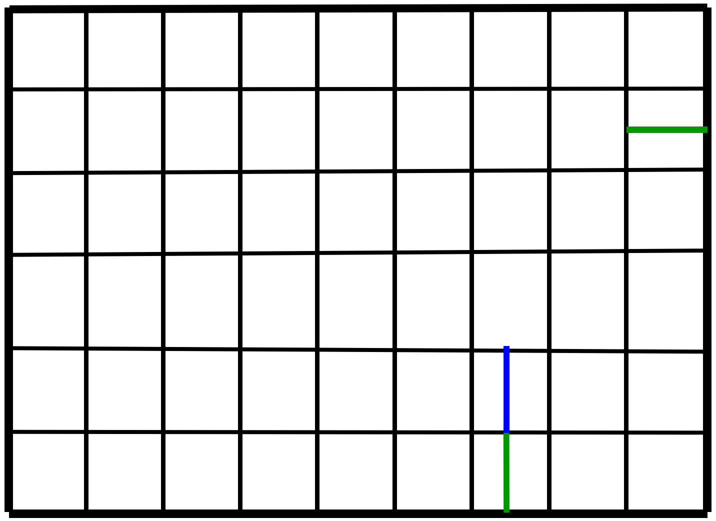

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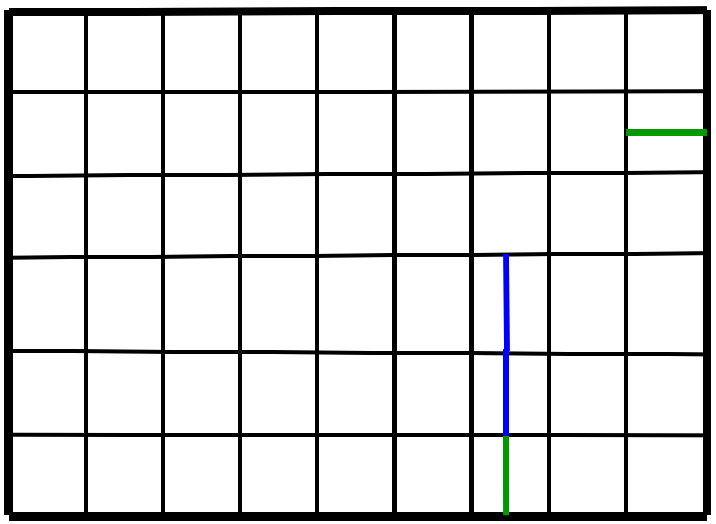

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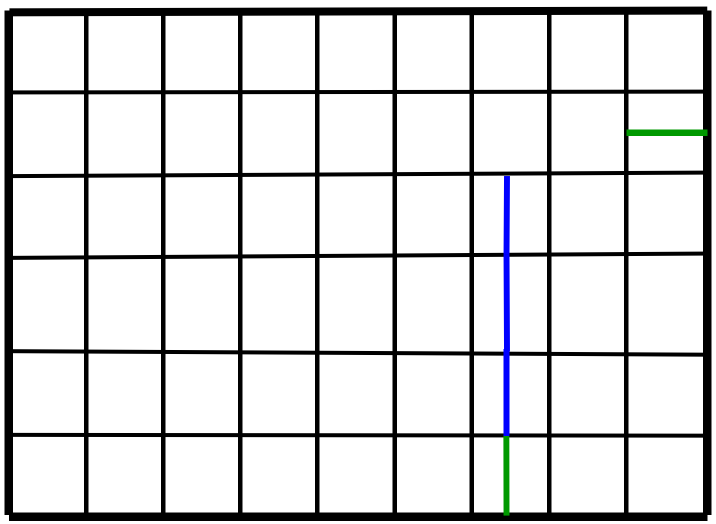

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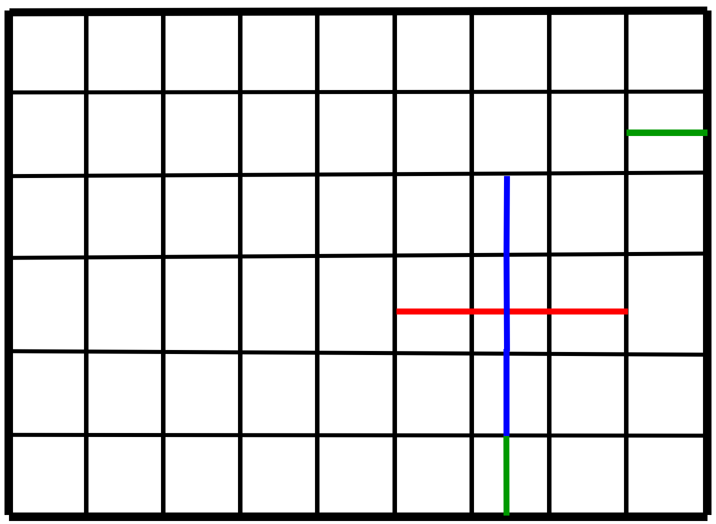

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Interior mesh-rectangle length 4, T-joints at both ends. Dimension increase 1.

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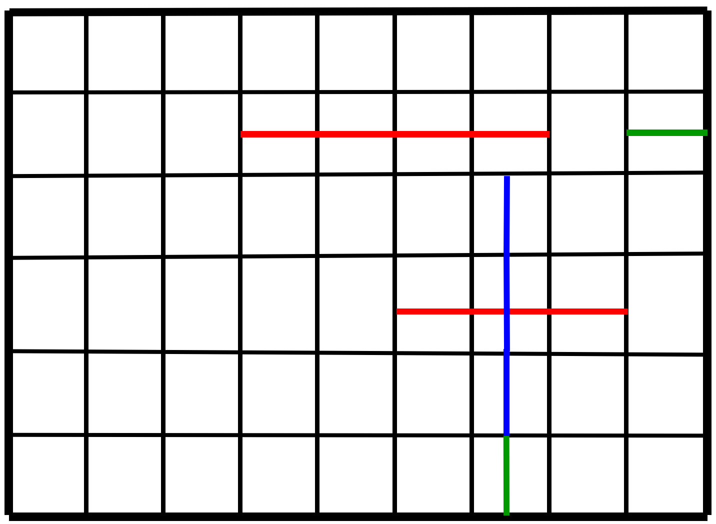

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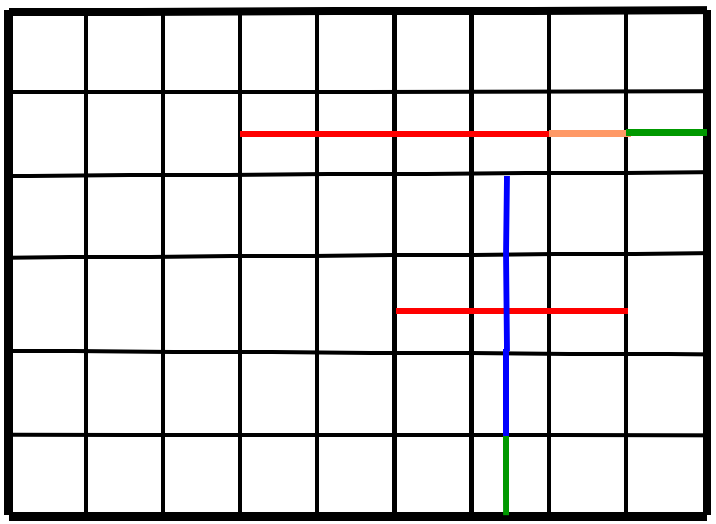

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Mesh-rectangle length 1 gap filling. Dimension increase 4, $\mathcal{B}_\gamma$ spans a polynomial space

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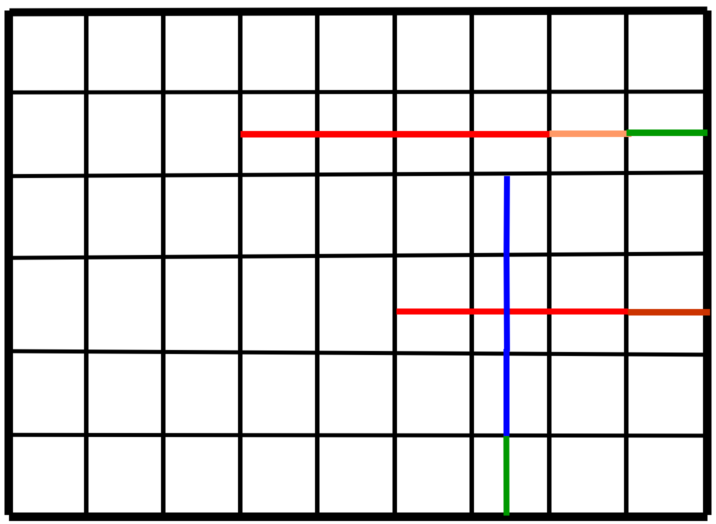

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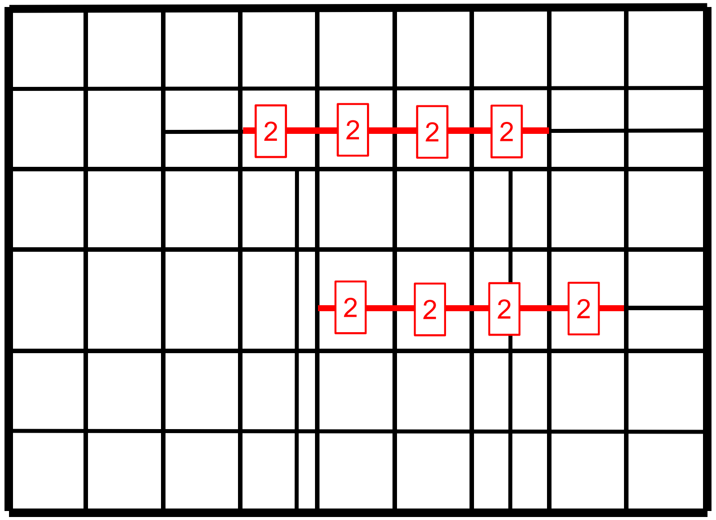
Mesh-rectangle length 1 gap filling. Dimension increase 4, $\mathcal{B}_\gamma$ spans a polynomial space

Mesh-rectangle length 1 extension of existing mesh-rectangle to the boundary. Dimension increase 4, $\mathcal{B}_\gamma$ spans a polynomial space

Increasing interior multiplicity in the bi-cubic case



Interior mesh-rectangle length 4, increase multiplicity to 2, lower multiplicity at both ends, dimension increase 1.

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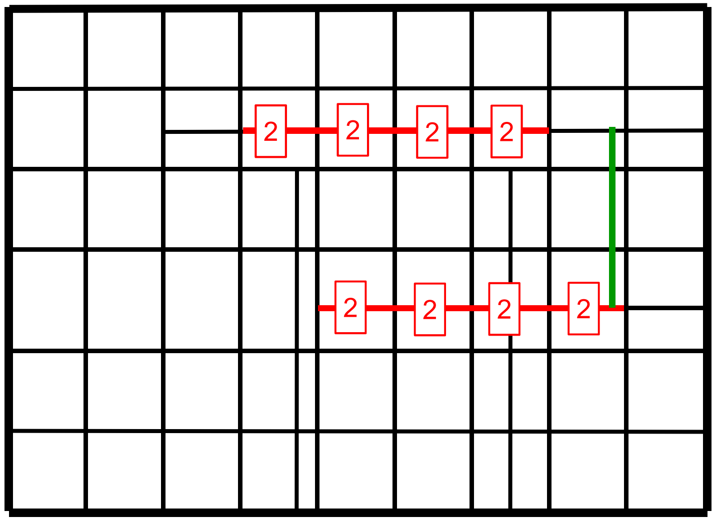

Interior mesh-rectangle length 4, increase multiplicity to 2, lower multiplicity at both ends, dimension increase 1.

Interior mesh-rectangle length 3, ending in T-joints with orthogonal mesh rectangles, one with multiplicity 1, and one with multiplicity 2, dimension increase 1.

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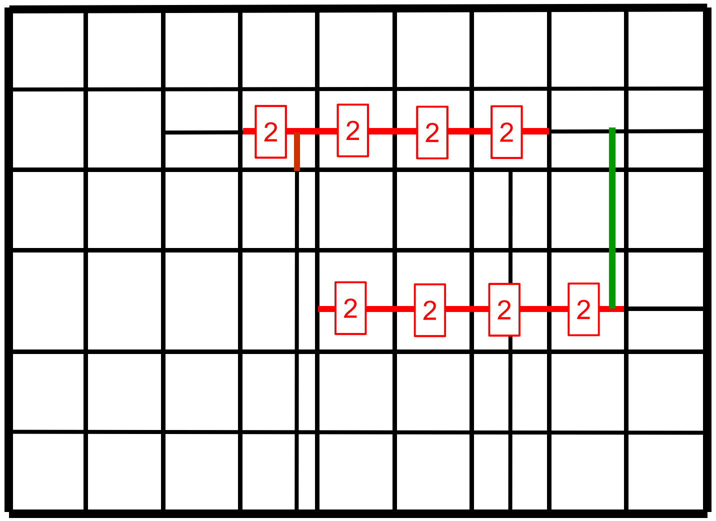

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Interior mesh-rectangle length 3, ending in T-joints with orthogonal mesh rectangles, one with multiplicity 1, and one with multiplicity 2, dimension increase 1.

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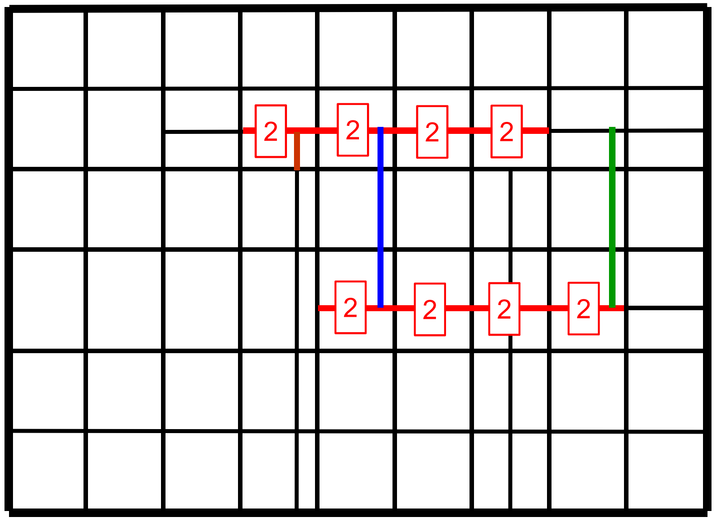

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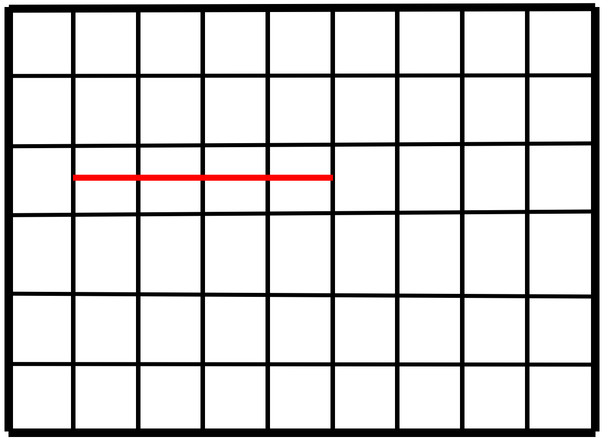
Interior mesh-rectangle length 3, ending in T-joints with orthogonal mesh rectangles, one with multiplicity 1, and one with multiplicity 2, dimension increase 1.

Extend existing mesh by length 1, ending in T-joint with orthogonal mesh rectangles with multiplicity 2, dimension increase 2, $\mathcal{B}_\gamma$ spans a polynomial space.

Interior mesh-rectangle length 3, ending in T-joints with orthogonal mesh rectangles of multiplicity 2,, dimension increase 2. To decide if $\mathcal{B}_{j+1}$ is a basis check if $\dim \operatorname{span} \left(B_\gamma\right)_{B\in\mathcal{B}_\gamma} = 2$.

Possible to increase dimension without refining LR B-splines (violation of LR B-spline refinement rule)



Dimension increase 1, one new B-splines (+5, -4)

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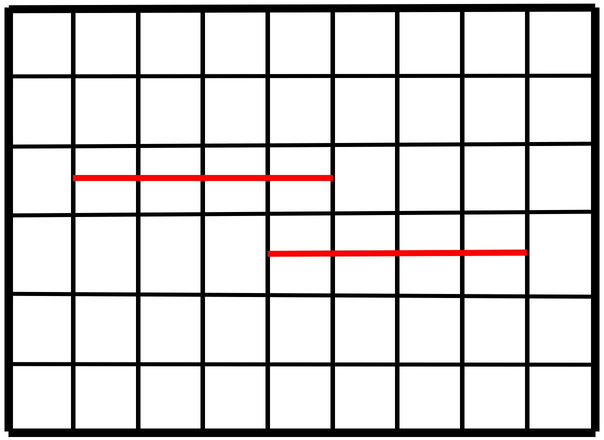

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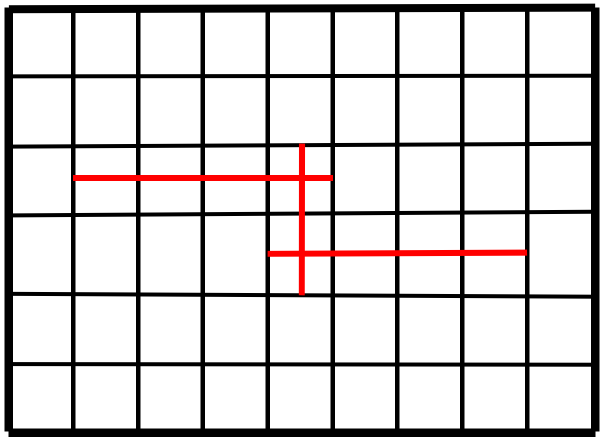

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Dimension increase 1, no new B-splines

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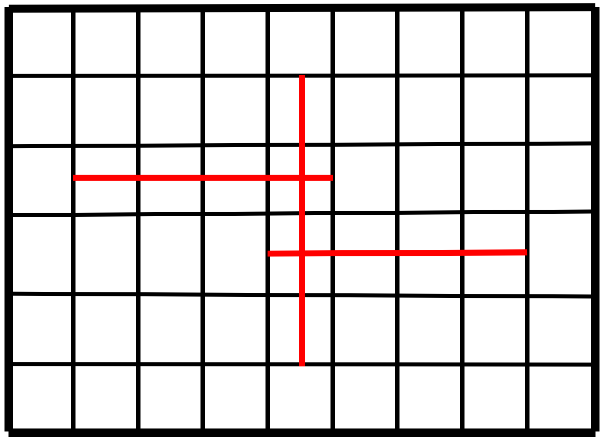

Dimension increase 1, one new B-splines (+5, -4)

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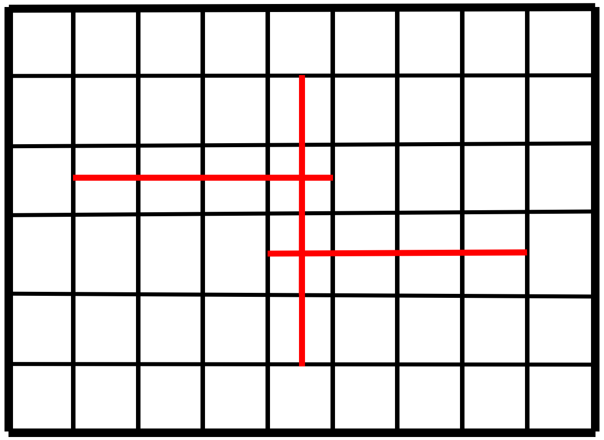
Dimension increase 3, three new B-splines (+ 9, -6)

- To decide if $\mathcal{B}_{j+1}$ is a basis check if $\dim \operatorname{span} \left(B_\gamma\right)_{B \in \mathcal{B}_\gamma} = 3.$

Possible to increase dimension without refining LR B-splines (violation of LR B-spline refinement rule)



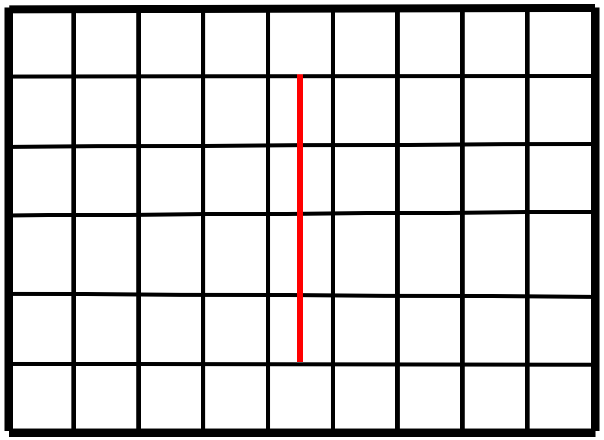
Dimension increase 1, one new B-splines (+5, -4)

Dimension increase 1, one new B-splines (+5, -4)

Dimension increase 1, no new B-splines

Dimension increase 3, three new B-splines (+ 9, -6)

- To decide if $\mathcal{B}_{j+1}$ is a basis check if $\dim \operatorname{span} \left(B_\gamma\right)_{B \in \mathcal{B}_\gamma} = 3.$

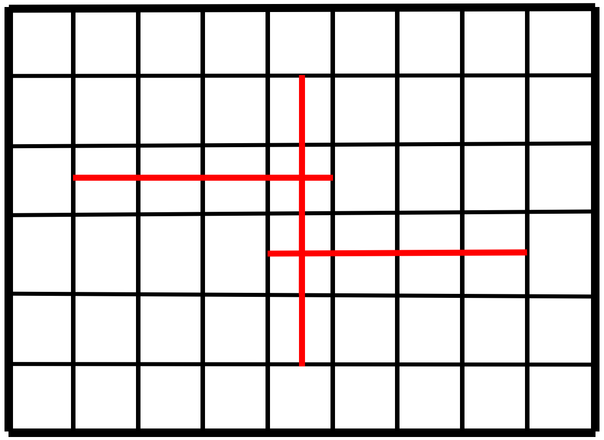


Alternative refinement sequence

Dimension increase 1, one new B-spline (+5, -4)

Dimension increase 1, one new B-spline (+2, -1)

Possible to increase dimension without refining LR B-splines (violation of LR B-spline refinement rule)



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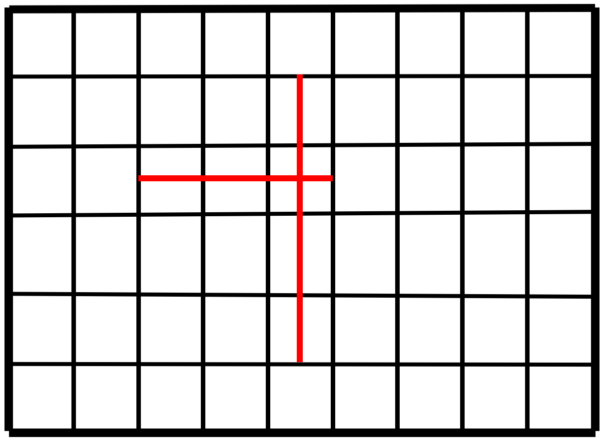
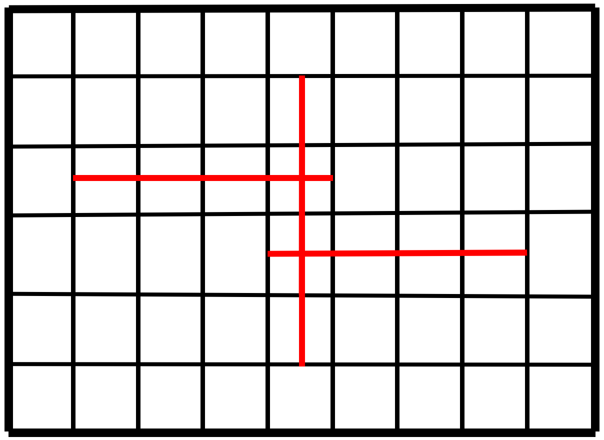
Alternative refinement sequence



Dimension increase 1, one new B-spline (+2, -1)

Possible to increase dimension without refining LR B-splines (violation of LR B-spline refinement rule)



Dimension increase 1, one new B-splines (+5, -4)

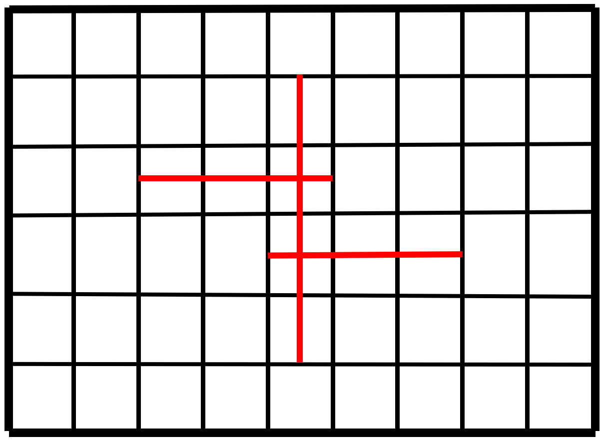
Dimension increase 1, one new B-splines (+5, -4)

Dimension increase 1, no new B-splines

Dimension increase 3, three new B-splines (+ 9, -6)

- To decide if $\mathcal{B}_{j+1}$ is a basis check if $\dim \operatorname{span} \left(B_\gamma\right)_{B \in \mathcal{B}_\gamma} = 3.$

Alternative refinement sequence

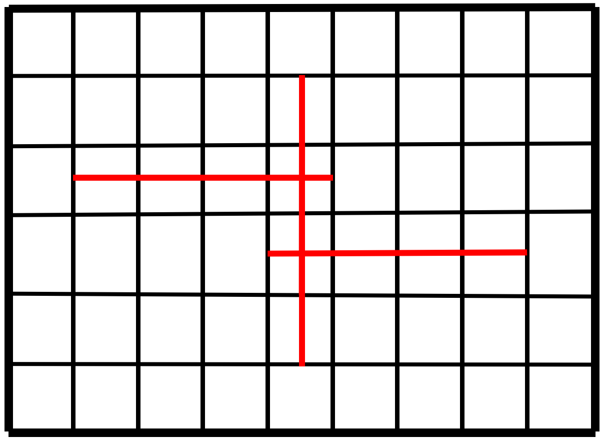


Dimension increase 1, one new B-spline (+5, -4)

Dimension increase 1, one new B-spline (+2, -1)

Dimension increase 1, one new B-spline (+2, -1)

Possible to increase dimension without refining LR B-splines (violation of LR B-spline refinement rule)



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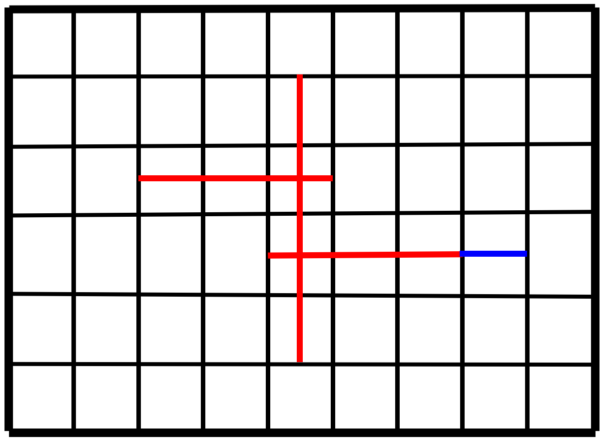
Dimension increase 1, no new B-splines

Dimension increase 3, three new B-splines (+ 9, -6)

- To decide if $\mathcal{B}_{j+1}$ is a basis check if $\dim \operatorname{span} \left(B_{\gamma}\right)_{B \in \mathcal{B}_{\gamma}} = 3.$

Alternative refinement sequence



Dimension increase 1, one new B-spline (+5, -4)

Dimension increase 1, one new B-spline (+2, -1)

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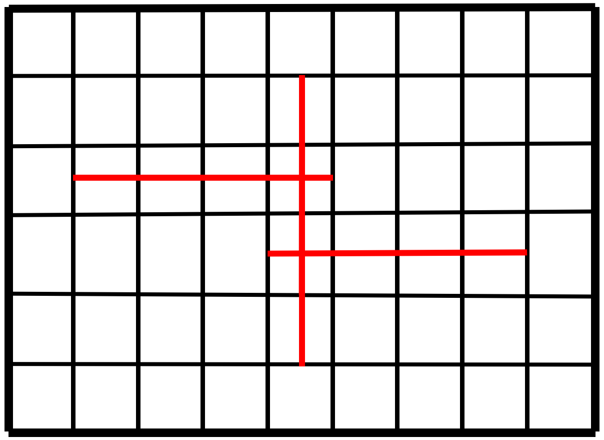

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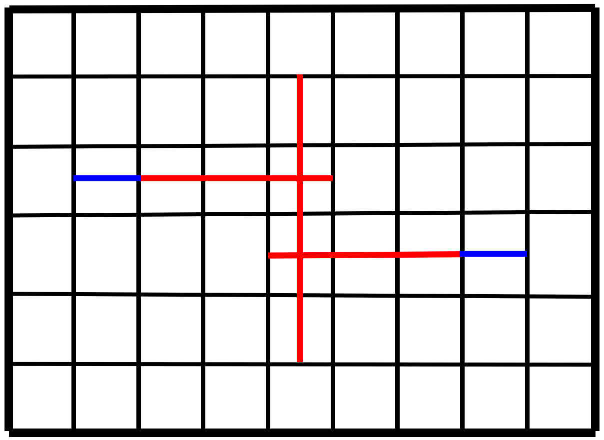
- To decide if $\mathcal{B}_{j+1}$ is a basis check if $\dim \operatorname{span} \left(B_\gamma\right)_{B\in\mathcal{B}_\gamma} = 3.$

Alternative refinement sequence



Dimension increase 1, one new B-spline (+5, -4)

Dimension increase 1, one new B-spline (+2, -1)

Dimension increase 1, one new B-spline (+2, -1)

Dimension increase 1, one new B-spline (+5, -4)

Dimension increase 1, one new B-spline (+5, -4)

What if $\#\mathcal{B}_{j+1} > \dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}_{j+1}, \mu_{j+1})$, e.g., linear dependence.

- Testing in the bi-cubic case shows that this can happen.
  - In examples run in 0.01% of the tested cases.
- What to do?
  - Discard refinement and try another refinement near by
  - Eliminate extra B-splines

Ensure linear independence in 2-variate case

- Formula for increase in the dimension 2-variate case

$$\dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}+\gamma,\mu_\gamma) = \dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M},\mu) + \sum_{i=1}^{n} \tilde{\mu}_i - p - 1 - \Delta h_1 + \Delta h_0$$

  - $\tilde{\mu}_i, i = 1,..,n$, multiplicity of intersection points of γ and orthogonal mesh-rectangles, except if $\tilde{\mu}_i, = p+1,\ i = 1, n$ if γ is extension of existing meshrectangle/multiplicity.
  - $\Delta h_1, \Delta h_0$ always zero for LR-splines
- For dimension increase more than 1 compare dimension of $\mathcal{B}_\gamma$ with above increase to check for hand-in-hand
- Confirm that number of B-splines after refinement corresponds to $\dim \mathbb{S}_{\boldsymbol{p}}(\mathcal{M}+\gamma,\mu_\gamma)$.
- Can easily be checked for all refinements

Final remarks

- Linear independence of LR B-splines can be ensured by ensuring that the refinement goes hand-in-hand and check that the number of B-splines corresponds to the spline space.
  - The restriction refined B-splines to the refining mesh-rectangle provides an approach for checking the hand-in-hand property
  - Refinement should be a sequence of refinements with minimal dimension increase
  - In the 2-variate case minimal refinements results in either
    - Dimension increase by 1
    - Checking the dimension of a univariate polynomial space
    - In the cases of multiplicity higher than 1 the dimension of a univariate spline space possibly has to be established, e.g., by knot insertion and checking the rank of the knot insertion matrix.